On p-adic Siegel modular forms of non-real Nebentypus of degree 2

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Abstract

We show that all Siegel modular forms of non-real Nebentypus for $\Gamma_0^{(2)}(p)$ are p-adic Siegel modular forms by using a Maass lift.

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1 Introduction

In [9], Serre defined the notion of p-adic modular forms and applied it to the construction of a p-adic L-function. Recently, several people attempted to generalize this notion to that of the case of several variables. In particular, Böcherer-Nagaoka [1] defined the p-adic Siegel modular forms and showed that all Siegel modular forms with level p and real Nebentypus are p-adic Siegel modular forms. The aim of this paper is to generalize it to the case of non-real Nebentypus.

We state our results more precisely. Let k be a positive integer, p an odd prime and χ a Dirichlet character modulo p with $\chi(-1)=(-1)^k$. For the congruence subgroup $\Gamma_0^{(n)}(p)$ of the symplectic group $\Gamma_n = Sp_n(\mathbb{Z})$, we denote by $M_k(\Gamma_0^{(n)}(p),\chi)$ the space of corresponding Siegel modular forms of weight k and character χ. For a subring R of $\mathbb{C}$, let $M_k(\Gamma_0^{(n)}(p),\chi)_R \subset M_k(\Gamma_0^{(n)}(p),\chi)$ denote the R-module of all modular forms whose Fourier coefficients belong to R. Let μ_{p-1} denote the group of the $(p-1)$-th roots of unity in $\mathbb{C}^\times$. We fix an embedding σ from $\mathbb{Q}(\mu_{p-1})$ to $\mathbb{Q}_p$ (see Subsection 2.4). The following theorem is our main result:

Theorem 1.1. *For any modular form $F \in M_k(\Gamma_0^{(2)}(p),\chi)_{\mathbb{Q}(\mu_{p-1})}$, F^σ is a p-adic Siegel modular form. In other words, there exists a sequence of full modular forms $\{G_{k_m}\}$ such that*

$$\lim_{m\to\infty} G_{k_m} = F^\sigma \quad (p\text{-}adically).$$

In Section 3, we prove Theorem 1.1. The key point of the proof is the following existence theorem: Let ω be the Tichmüler character on $\mathbb{Z}_p$.

Theorem 1.2. *We take $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $\chi^\sigma = \omega^\alpha$. Then there exists a sequence of modular forms $\{G_{k_m} \in M_{k_m}(\Gamma_0^{(2)}(p),\chi)_{\mathbb{Q}(\mu_{p-1})}\}$ such that*

$$\lim_{m\to\infty} G_{k_m}^\sigma = 1 \quad (p\text{-}adically).$$

Remark 1.3. If we denote by $\boldsymbol{X} := \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$ the group of the weights of p-adic Siegel modular forms, the sequence $\{k_m\}$ of weights in Theorem 1.2 converges automatically $(0,-\alpha)$ in $\boldsymbol{X}$ by the results [2, 6, 9].

2 Preliminaries

2.1 Siegel modular forms

Let $\mathbb{H}_n$ be the Siegel upper-half space of degree n. The Siegel modular group $\Gamma_n = Sp_n(\mathbb{Z})$ acts on $\mathbb{H}_n$ by the generalized fractional transformation

$$MZ := (AZ+B)(CZ+D)^{-1}, \quad for\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n.$$

Let N be a positive integer. The congruence subgroup $\Gamma_0^{(n)}(N)$ is defined by

$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \middle| C \equiv O_n \bmod N \right\}.$$

Let χ be a Dirichlet character modulo N. The space $M_k(\Gamma_0^{(n)}(N), \chi)$ of Siegel modular forms of weight k and character χ consists of all of holomorphic functions $f : \mathbb{H}_n \to \mathbb{C}$ satisfying

$$f(MZ) = \chi(\det D)\det(CZ+D)^k f(Z), \quad for\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N).$$

If χ is trivial, we write as $M_k(\Gamma_0^{(n)}(N)) = M_k(\Gamma_0^{(n)}(N), \chi)$ simply. If $f \in M_k(\Gamma_0^{(n)}(N), \chi)$ then f has a Fourier expansion of the form

$$f = \sum_{O \leq T \in \Lambda_n} a_f(T) e^{2\pi i \mathrm{tr}(TZ)},$$

where T runs over all elements of semi-positive definite of Λ_n and

$$\Lambda_n := \{T = (t_{ij}) \in Sym_n(\mathbb{Q}) | t_{ii} \in \mathbb{Z},\ 2t_{ij} \in \mathbb{Z}\}.$$

In this paper, we mainly deal with the case where N is a prime.

2.2 p-adic Siegel modular forms

Let v_p be the normalized additive valuation on $\mathbb{Q}_p$ as $v_p(p) = 1$. We consider a formal power series of the form $f = \sum_{O \leq T \in \Lambda_n} a(T) e^{2\pi i \mathrm{tr}(TZ)}$ with $a(T) \in \mathbb{Q}_p$. For more accurate interpretation of f, see [1, 2].

Definition 2.1. A formal power series $f = \sum_{O \leq T \in \Lambda_n} a(T) e^{2\pi i \mathrm{tr}(TZ)}$ with $a(T) \in \mathbb{Q}_p$ called a *p-adic Siegel modular form* if there exists a sequence of full modular forms $\{g_m\} \subset M_{k_m}(\Gamma_2)_{\mathbb{Q}}$ such that $\lim_{m \to \infty} g_m = f$ (p-adically), where the limit means that $\inf_{T \in \Lambda_n}(v_p(a_{g_m}(T) - a(T))) \to \infty$ as $m \to \infty$.

In [1], Böcherer and Nagaoka showed that

Theorem 2.2 (Böcherer-Nagaoka [1])**.** Let p be an odd prime. If $f \in M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}}$ then f is a p-adic Siegel modular form.

2.3 Jacobi forms and their liftings

In this subsection, we recall the known facts related Jacobi forms and their liftings. Since we do not need the general level case, we only consider the prime level case.

Let p be an odd prime and χ a Dirichlet character modulo p with $\chi(-1)=(-1)^k$. Let ϕ be a Jacobi form of weight k, index 1 and character χ with respect to $\Gamma_0^{(1)}(p)$. Then ϕ has a Fourier expansion of the form

$$\phi(\tau,z)=\sum_{n=0}^{\infty}\sum_{\substack{r\in\mathbb{Z}\\ 4n-r^2\geq 0}} c(n,r)q^n\zeta^r, \quad for\ (\tau,z)\in\mathbb{H}_1\times\mathbb{C},$$

where $q:=e^{2\pi i\tau}$ and $\zeta:=e^{2\pi iz}$. The Maass lift $\mathcal{M}\phi\in M_k(\Gamma_0^{(2)}(p),\chi)$ of ϕ is described by

$$\begin{aligned}\mathcal{M}\phi(Z)=&\left(\frac{1}{2}L(1-k,\chi)+\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^{k-1}q^n\right)c(0,0)\\ &+\sum_{l=1}^{\infty}\sum_{4nl-r^2\geq 0}\sum_{\substack{d|(n,r,l)\\ (d,p)=1}}\chi(d)d^{k-1}c\left(\frac{nl}{d^2},\frac{r}{d}\right)q^n\zeta^r q'^l, \quad for\ Z=\begin{pmatrix}\tau & z\\ z & w\end{pmatrix}\in\mathbb{H}_2,\end{aligned}$$

where $q':=e^{2\pi iw}$. This lift was studied by Ibukiyama. For the precise definitions of Jacobi forms with level and their liftings, see [5, 7].

2.4 Embeddings from $\mathbb{Q}(\mu_{p-1})$ to $\mathbb{Q}_p$

In this subsection, we mention that how to determine the embeddings from $\mathbb{Q}(\mu_{p-1})$ to $\mathbb{Q}_p$.

Let μ_{p-1} denote the group of the $(p-1)$-th roots of unity in $\mathbb{C}^\times$. Let us take a generator ζ_{p-1} of μ_{p-1} and consider the prime ideal factorization of p in the ring $\mathbb{Z}[\zeta_{p-1}]$ of integers of $\mathbb{Q}(\mu_{p-1})$. Let $\Phi(X)\in\mathbb{Z}[X]$ be the minimal polynomial of ζ_{p-1}, namely $\Phi(X)$ is the cyclotomic polynomial having the root ζ_{p-1}. We can always decompose $\Phi(X)$ as the form $\Phi(X)\equiv q_1(X)\cdots q_r(X) \bmod p$, where $r=\varphi(p-1)$, each $q_i(X)$ is a polynomial of degree one with $q_i(X)\not\equiv q_j(X) \bmod p$. Then p is decomposed as a product of r prime ideals $\mathfrak{p}_i:=(q_i(\zeta_{p-1}),p)$, namely we have the perfect decomposition

$$(p)=\mathfrak{p}_1\cdots\mathfrak{p}_r=(q_1(\zeta_{p-1}),p)\cdots(q_r(\zeta_{p-1}),p).$$

If we write $q_i(X)=X-d_i$ for some $d_i\in\mathbb{Z}$, then an embedding σ_i from $\mathbb{Q}(\zeta_{p-1})$ to $\mathbb{Q}_p$ corresponding $\mathfrak{p}_i$ is determined by $\sigma_i(\zeta_{p-1})=\omega(d_i)$.

Example 2.3. (1) **Case** $p=5$ ($\zeta_4=i$).
We see easily that $\Phi(X)=X^2+1\equiv(X-2)(X-3) \bmod 5$. Putting $\mathfrak{p}_1:=(i-2,5)$ and $\mathfrak{p}_2:=(i-3,5)$, then $(5)=\mathfrak{p}_1\mathfrak{p}_2$. In fact, $(i-2,5)=(i-2)$ and $(i-3,5)=(i+2)$. Hence, the embeddings σ_i corresponding $\mathfrak{p}_i$ are determined by $\sigma_1(i)=\omega(2)$ and $\sigma_2(i)=\omega(3)$.
(2) **Case** $p=7$ ($\zeta_6=(1+\sqrt{3}i)/2$).
One has $\Phi(X)=X^2-X+1\equiv(X-3)(X-5) \bmod 7$. If we set $\mathfrak{p}_1:=(\zeta_6-3,5)$ and $\mathfrak{p}_2:=(\zeta_6-5,5)$, then $7=\mathfrak{p}_1\mathfrak{p}_2$. Hence, the embedding σ_i are determined by $\sigma_1(\zeta_6)=\omega(3)$ and $\sigma_2(\zeta_6)=\omega(5)$.

3 Proofs

In this section, we prove our theorems. As introduced in Remark 1.3, let $\boldsymbol{X}:=\mathbb{Z}_p\times\mathbb{Z}/(p-1)\mathbb{Z}$ denote the group of p-adic Siegel modular forms. Following Serre's notation in [9], let us write $\zeta^*(s,u):=L_p(s,\omega^{1-u})$ for $(s,u)\in\boldsymbol{X}$, where $L_p(s,\chi)$ is the Kubota-Leopoldt's p-adic L-function with character χ (e.g. [4]).

3.1 Proof of Theorem 1.2

We take a sequence $\{k_m = ap^m\}$ for $0 < a \in \mathbb{Z}$ with $a \equiv -\alpha \bmod p-1$. Note that a is even or odd according as χ is even or odd.

As in [3], let $E^J_{k,1}(\tau, z)$ be the normalized Siegel Jacobi Eisenstein series of weight k and index 1 (i.e. the constant term is 1). It is known that its Fourier coefficients are in $\mathbb{Q}$. Moreover we denote by

$$E^{(1)}_{k,\chi} = 1 + 2L(1-k,\chi)^{-1} \sum_{n=1}^{\infty} \sum_{0<d|n} \chi(d) d^{k-1} q^n \in M_k(\Gamma^{(1)}_0(p), \chi), \tag{3.1}$$

$$E^{(1)}_{k} = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sum_{0<d|n} d^{k-1} q^n \in M_k(\Gamma_1) \tag{3.2}$$

the normalized Eisenstein series of weight k for Γ_1 and normalized Hecke's Eisenstein series of weight k and character χ for $\Gamma^{(1)}_0(p)$, respectively. If we put

$$\phi_{k_m} := E^{(1)}_{a(p-2),\chi} E^{(1)}_{ap(p^{m-1}-1)} E^J_{2a,1}$$

then we see that ϕ_{k_m} is a Jacobi form of weight k_m and index 1 with character χ for $\Gamma^{(1)}_0(p)$. Here note that $E^{(1)}_{ap(p^{m-1}-1)} E^J_{2a,1}$ has rational Fourier coefficients. Moreover if we write its Fourier expansion as $\phi_{k_m} = \sum_{n,r} c_{k_m}(n,r) q^n \zeta^r$, then $c_{k_m}(n,r) \in \mathbb{Q}(\mu_{p-1})$. Now we can prove

Lemma 3.1. $\{\phi^{\sigma}_{k_m}\}$ converges in the formal power series ring $\mathbb{Q}_p[\![q,\zeta]\!]$. Namely, each coefficient $c_{k_m}(n,r)^{\sigma}$ converges in $\mathbb{Q}_p$.

Proof. Recall that

$$\phi^{\sigma}_{k_m} = (E^{(1)}_{a(p-2),\chi} E^{(1)}_{ap(p^{m-1}-1)} E^J_{2a,1})^{\sigma} = (E^{(1)}_{a(p-2),\chi})^{\sigma} E^{(1)}_{ap(p^{m-1}-1)} E^J_{2a,1} \in \mathbb{Q}_p[\![q,\zeta]\!].$$

Hence we may only show that $\lim_{m\to\infty} E^{(1)}_{ap(p^{m-1}-1)} \in \mathbb{Q}_p[\![q]\!]$. To prove this, we consider the Eisenstein series

$$G^{(1)}_{l_m} := -\frac{B_{l_m}}{2l_m} E^{(1)}_{l_m} = -\frac{B_{l_m}}{2l_m} + \sum_{n=1}^{\infty} \sum_{0<d|n} d^{l_m-1} q^n,$$

where we put $l_m := ap(p^{m-1}-1)$ for the sake of simplicity. It is clear that $\{l_m\}$ is a Cauchy sequence. Hence there exists a limiting value $\lim_{m\to\infty} \sum_{0<d|n} d^{l_m-1} \in \mathbb{Q}_p$ for every $n \geq 1$. Since l_m tends to $(-ap, 0) \neq (0,0)$ in $\boldsymbol{X}$, we can apply Corollaire 2 in [9] to $G^{(1)}_{l_m}$. Therefore we see that the constant term also converges in $\mathbb{Q}_p$, namely

$$-\lim_{m\to\infty} \frac{B_{l_m}}{2l_m} \in \mathbb{Q}_p.$$

Now we shall show that this value is not zero. If $m \geq 2$ then $p-1 | l_m$. Hence the denominator of B_{l_m} is divisible by p according to Von-Staudt Clausen theorem. Moreover $p \| l_m$. Summarizing these facts, we see that the denominator of $B_{l_m}/2l_m$ is divisible by p^2 for every $m \geq 2$. It follows immediately from this property that

$$-\lim_{m\to\infty} \frac{B_{l_m}}{2l_m} \neq 0.$$

Therefore we get

$$\lim_{m\to\infty} E^{(1)}_{l_m} = \lim_{m\to\infty} \left(1 - \frac{2l_m}{B_{l_m}} \sum_{n=1}^{\infty} \sum_{0<d|n} d^{l_m-1} q^n \right) \in \mathbb{Q}_p[\![q]\!].$$

This completes the proof of Lemma 3.1. $\square$

Let us return to the proof of Theorem 1.2. Taking the Maass lift $\mathcal{M}\phi_{k_m} =: G_{k_m} \in M_{k_m}(\Gamma_0^{(2)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}$, we have the following Fourier expansion

$$\begin{aligned} G_{k_m} &= \frac{1}{2}L(1-k_m,\chi) + \sum_{n=1}^{\infty}\sum_{0<d|n}\chi(d)d^{k_m-1}(n)q^n \\ &+ \sum_{l=1}^{\infty}\sum_{4nl-r^2\geq 0}\sum_{\substack{0<d|(n,r,l)\\(p,d)=1}}\chi(d)d^{k_m-1}c_{k_m}\left(\frac{nl}{d^2},\frac{r}{d}\right)q^n\zeta^r q'^l. \end{aligned}$$

The $l>0$-th Fourier Jacobi coefficient is

$$\sum_{4nl-r^2\geq 0}\sum_{\substack{0<d|(n,r,l)\\(p,d)=1}}\chi(d)d^{k_m-1}c_{k_m}\left(\frac{nl}{d^2},\frac{r}{d}\right)q^n\zeta^r.$$

Since $\chi(d)^\sigma = \omega(d)^\alpha = d^\alpha$, if we take σ, then

$$\sum_{4nl-r^2\geq 0}\sum_{\substack{0<d|(n,r,l)\\(p,d)=1}}d^{k_m+\alpha-1}c_{k_m}\left(\frac{nl}{d^2},\frac{r}{d}\right)^\sigma q^n\zeta^r.$$

The first Fourier Jacobi coefficient is Hecke's Eisenstein series of weight k_m and character χ in (3.1). By a similar argument of Serre, we obtain

$$\left(\frac{1}{2}L(1-k_m,\chi) + \sum_{n=1}^{\infty}\sum_{0<d|n}\chi(d)d^{k_m-1}q^n\right)^\sigma = \zeta^*(1-k_m,1-k_m-\alpha) + \sum_{n=1}^{\infty}\sum_{\substack{0<d|n\\(p,d)=1}}d^{k_m+\alpha-1}(n)q^n.$$

Finally, we set $G_{k_m} := 2L(1-k_m,\chi)^{-1}F_{k_m}$. Since k_m tends to $(0,-\alpha)$ in $\boldsymbol{X}$, $(k_m, k_m+\alpha)$ tends to $(0,0)$ in $\boldsymbol{X}$. Note that $\zeta^*(s,u)$ has a simple pole at $(1,1)$. Combining this fact with Lemma 3.1, we see that $G_{k_m}^\sigma$ tends to 1. In fact, the q-expansion of $G_{k_m}^\sigma$ is given by

$$\begin{aligned} G_{k_m}^\sigma &= 1 + \frac{1}{\zeta^*(1-k_m,1-k_m-\alpha)}\sum_{n=1}^{\infty}\sum_{\substack{0<d|n\\(p,d)=1}}d^{k_m+\alpha-1}q^n \\ &+ \frac{1}{\zeta^*(1-k_m,1-k_m-\alpha)}\left(\sum_{l=1}^{\infty}\sum_{4nl-r^2\geq 0}\sum_{\substack{0<d|(n,r,l)\\(p,d)=1}}d^{k_m+\alpha-1}c_{k_m}\left(\frac{nl}{d^2},\frac{r}{d}\right)^\sigma q^n\zeta^r q'^l\right). \end{aligned}$$

This completes the proof of Theorem 1.2. $\square$

3.2 Proof of Theorem 1.1

In order to apply Serre's argument, we start with proving that

Lemma 3.2. Let $f \in M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}(\mu_{p-1})}$. Then f is a $\mathbb{Q}(\mu_{p-1})$-linear combination of elements of $M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}}$.

Proof. It holds that $M_k(\Gamma_0^{(n)}(p))_{\mathbb{C}} = M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}} \otimes \mathbb{C}$ by Shimura's result [8]. This fact tells us that $f \in M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}}(\mu_{p-1})$ is uniquely written in the form $f = \sum_{i=1}^N c_i f_i$ for some $c_i \in \mathbb{C}$ and

$f_i \in M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}}$. For each $\tau \in Aut(\mathbb{C}/\mathbb{Q}(\mu_{p-1}))$, $f^\tau = \sum_{i=1}^N c_i^\tau f_i$ because each f_i has rational Fourier coefficients. On the other hand, since Fourier coefficients of f are in $\mathbb{Q}(\mu_{p-1})$, we have $f^\tau = f = \sum_{i=1}^N c_i f_i$. It follows from uniqueness of description of f that $c_i^\tau = c_i$. The assertion follows. □

We are now in a position to prove our main theorem.

Proof of Theorem 1.1. For any $F \in M_k(\Gamma_0^{(2)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}$, take a sequence of modular forms $\{G_{k_m} \in M_{k_m}(\Gamma_0^{(2)}(p), \chi^{-1})\}$ constructed in Theorem 1.2. We consider $FG_{k_m} \in M_{k+k_m}(\Gamma_0^{(2)}(p))_{\mathbb{Q}(\mu_{p-1})}$. Note here that each $k + k_m$ is even. Applying Lemma 3.2 to each FG_{k_m}, FG_{k_m} is a $\mathbb{Q}(\mu_{p-1})$-linear combination of elements of $M_{k+k_m}(\Gamma_0^{(2)}(p))_{\mathbb{Q}}$. Hence, $(FG_{k_m})^\sigma = F^\sigma G_{k_m}^\sigma$ is a p-adic Siegel modular form according to Theorem 2.2. Since $G_{k_m}^\sigma$ tends to 1, $F^\sigma G_{k_m}^\sigma$ tends to F^σ. Thus F^σ is a p-adic Siegel modular form. This completes the proof of Theorem 1.1. □

4 For generalization

In this section, we mention some remarks for generalization.

If the following problem is affirmative, then we can generalize Theorem 1.1 to the case of any degree.

Problem 4.1. Let k be a positive integer and p an odd prime. For any Dirichlet character χ modulo p with $\chi(-1) = (-1)^k$, we take $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $\chi^\sigma = \omega^\alpha$. Then, does there exist a sequence of Siegel modular forms $\{G_{k_m} \in M_{k_m}(\Gamma_0^{(n)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}\}$ such that

$$\lim_{m\to\infty} G_{k_m}^\sigma = 1 \quad (p\text{-}adically)?$$

Now we raise one more question which is equivalent to this problem.

Problem 4.2. Let p, χ and α be same as above. We take an integer a such that $a \equiv -\alpha \bmod p-1$. Then, does there exist a modular form $G_a \in M_a(\Gamma_0^{(n)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}$ such that

$$G_a^\sigma \equiv 1 \bmod p?$$

Remark 4.3. (1) If $\alpha = 0$ (i.e. $p-1|a$), then this problem is affirmative by Böcherer-Ngaoka's result.
(2) If this problem is affirmative, then we can solve Probem 4.1 affirmatively by putting $G_{k_m} := G_a^{p^m}$.

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